Signs of quadratic function

Part 4

Parametric equation

A parametric equation is an equation that is depending on a parameter m.

Example:

- $mx + 4 = 0$ is a parametric equation of first degree of main variable x and parameter m.
- $(t+1)x^2 + 2x - 3t = 0$ is a parametric equation of main variable x and of parameter t.

In this video, we will learn how to solve and discuss the existence of roots of a parametric equation in form of $ax^2 + bx + c = 0$

(as in the second example)

Parametric example #1

Consider the parametric equation:

$$(m+1)x^2 + 2x + m = 0 \; ; m \in IR$$

The coefficients are: $a = m + 1 \;\; ; b = 2 \; ; c = m$

Case 1: solve the equation for $a = 0$

- $a = 0 \;\; ; \;\; m + 1 = 0 \;\; ; \;\; m = -1$
- The equation becomes: $2x - 1 = 0$
- The solution is: $x = \frac{1}{2}$ single (unique) solution

Parametric example

Consider the parametric equation:

$$(m+1)x^2 + 2x + m = 0\ ;\ m \in IR$$

Case 2: solve the equation for $a \neq 0$

Step 1: Calculate the discriminant Δ $(or\ \Delta')$

$a \neq 0\ ;\ \ m \neq -1$

$\Delta' = b'^2 - ac = 1 - m(m+1) = 1 - m^2 - m = -m^2 - m + 1$

Parametric example

Consider the parametric equation:

$$(m+1)x^2 + 2x + m = 0\ ;\ m \in IR$$

Step 2: Discuss according to the values of m the signs of the discriminant

$\Delta' = -m^2 - m + 1$ Quadratic function

$$\delta_m = b^2 - 4ac = (-1)^2 - 4(-1)(1) = 1 + 4 = 5 > 0$$

$$m_1 = \frac{1-\sqrt{5}}{-2} = \frac{-1+\sqrt{5}}{2} \quad ; \quad m_2 = \frac{1+\sqrt{5}}{-2} = \frac{-1-\sqrt{5}}{2}$$

m	$-\infty$	m_2		-1		m_1	$+\infty$
Δ'	$-$	0	$+$	‖	$+$	0	$-$
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Parametric example

Consider the parametric equation:

$$(m+1)x^2 + 2x + m = 0\ ;\ m \in IR$$

Step 3: Discuss the existence and the number of solutions of the equation

m	$-\infty$		m_2		-1		m_1		$+\infty$
Δ'		$-$	0	$+$	‖	$+$	0	$-$	

- If $m \in]-\infty; m_2[\cup]m_1; +\infty[$:

  no real roots

- If $m \in]m_2; m_1[$:

2 distinct roots $x = \frac{-b' \pm \sqrt{\Delta}}{a} = \frac{-1 \pm \sqrt{-m^2 - m + 1}}{m+1}$

- If $m \in \{m_1; m_2\}$

1 double root $x_1 = x_2 = -\frac{b'}{a} = \frac{-2}{m+1} = \begin{cases} \frac{-4}{1+\sqrt{5}} & if\ m = m_1 \\ \frac{-4}{1-\sqrt{5}} & if\ m = m_2 \end{cases}$

Parametric example #2

Consider the parametric equation:

$$(1+m^2)x^2 + 2\sqrt{3}x - 1 = 0 \; ; m \in IR$$

The coefficients are: $a = 1 + m^2 \; ; b = 2\sqrt{3} \; ; c = -1$

$a = 1 + m^2 \neq 0$ **for all values of** m

$\Delta' = 3 - (-1)(1+m^2) = 3 + 1 + m^2 = 4 + m^2 > 0$ for all values of m.

The equation has two distinct roots for all values of m:

$$x = \frac{-b' \pm \sqrt{\Delta'}}{a} = \frac{-\sqrt{3} \pm \sqrt{4+m^2}}{1+m^2}$$

Parametric example #3

Consider the parametric equation:

$$x^2 + 2x + m^2 + 1 = 0 \; ; m \in IR$$

The coefficients are: $a = 1 \; ; b = 2 \; ; c = m^2 + 1$

$a = 1 \neq 0$ **for all values of** m

$\Delta' = 1 - (1)(1 + m^2) = -m^2 = -m^2 \leq 0$ for all values of m.

- If $m = 0 \; ; \; \Delta' = 0$

The equation has one double root: $x_1 = x_2 = -\frac{b'}{a} = -\frac{1}{1} = -1$

- If $m \neq 0 \; ; \; \Delta' < 0$

The equation has no real roots

Now it is your turn

Can you solve this parametric equation????

$$x^2 + (3m - 4)x + 4 = 0; m \epsilon IR$$